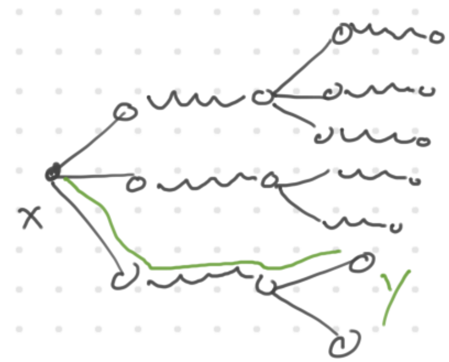
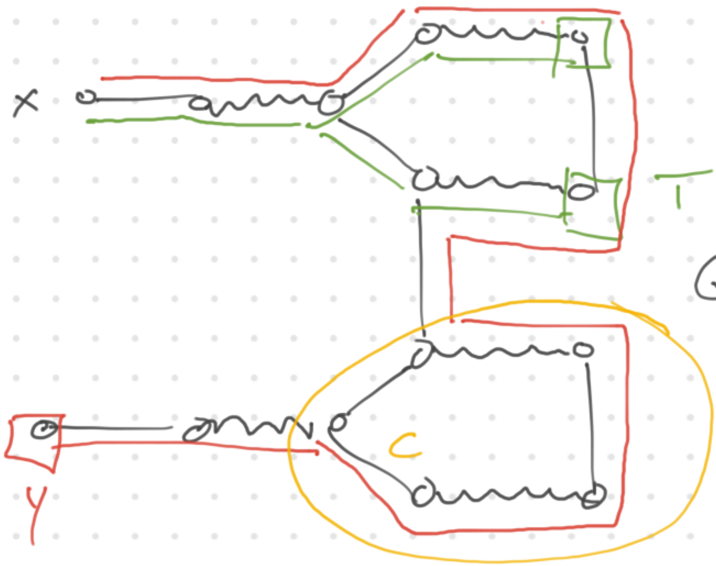
- maximum cardinality matching in bipartite graphs - grow M-alternating BFS tree
- max weight perfect matching in bipartite graphs - Hungarian algorithm
- Maximum size matchings in general graphs - Edmond's blossom algorithm (1961)

<u>Prop</u> M a matching in a graph G $\exists$ M' a matching $|M'| > |M|$ $\iff$ $\exists$ an M-augmenting path.

<u>Prop</u> M a matching in G. G bipartite. Let T be the M-alt. BFS tree w/ root x. Then $\exists$ an M-augmenting path w/ x as an endpoint $\iff$ $\exists$ M-augmenting path γ in T from $x \to$ a leaf.



Prop is NOT true in general graphs



M-alt. BFS tree

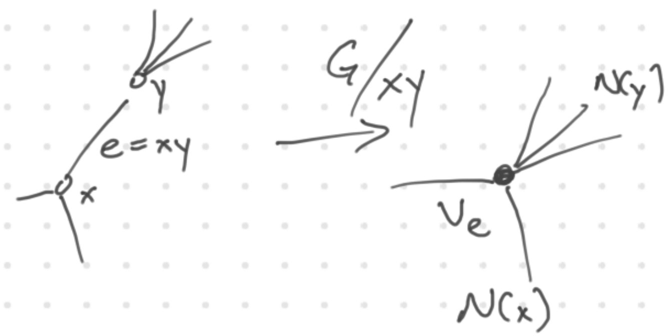
G + it does not contain an M-augmenting path with ~~xxx~~ x as an endpoint.

But an M-augmenting path DOES exist.

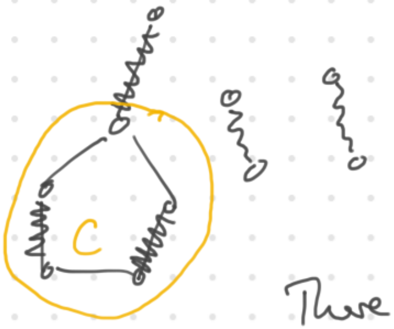
We do have an odd cycle C forming blossom

Def M a matching in a graph G, ~~a~~ A blossom is an odd cycle C of length $2k+1$ st $|E(C) \cap M| = k$.

Def to contract an edge e in a graph G, we identify the two ends



Let G be a graph, M a matching, + C an M-blossom

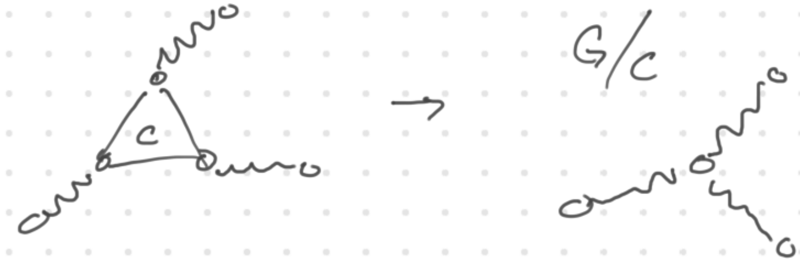


There is at most *one* matching edge "leaving" C

ie ≤ 1 edge of M w/ exactly one endpoint in C.

$\Rightarrow$ OBS $M - E(C)$ is a matching in G/C.

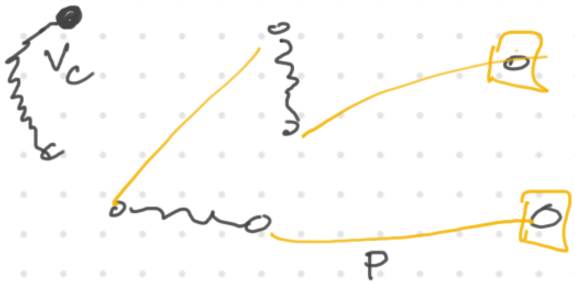
NOT True if C is not a blossom



+ $M - E(C)$ is not a matching.

Prop M a matching in G, C an M blossom. if $\exists$ a matching $\overline{M}$ which is strictly larger than $|M - E(C)|$ in G/C, Then $\exists$ a matching M' in G w/ $|M'| > |M|$

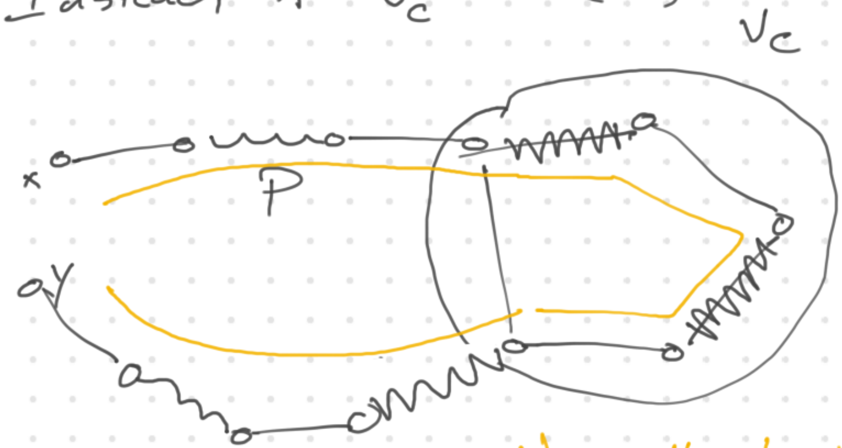
pf Assume $\exists$ $\overline{M}$ in G/C strictly larger than $E(M) - E(C)$. Then $\exists$ an $(E(M) - E(C))$ - augmenting path in G/C - let v_c be the vertex of G/C corresponding to the contracted cycle C.



Let P be ~~a~~ such an augmenting path.

Then P either uses the vertex v_c or not. If $v_c \notin V(P)$, then P is a ~~...~~ path in G as well which is an M-alternating path connecting 2 unmatched vertices ie M-augmenting path.

Instead, if $v_c \in V(P)$.



No matter ~~to~~ which vertex of C leads to x, we can always route one way around ~~to~~ the odd cycle C to get an augmenting path.

we conclude that $\exists$ an M-augmenting path in G $\Rightarrow$ $\exists$ M' a matching w/ $|M'| > |M|$.

This the basis for a recursive algorithm
Starting with a matching M, either find an M-augmenting path or determine that M is maximum.

Step 1: subroutine that either finds
- M-augmenting path
- determines M is maximal
- or finds an M-blossom.

Step 2: if we find an M-blossom C we recurse on the graph G/C & matching $M - E(C)$

<u>Step 3</u> considers output of recursion: either we find an

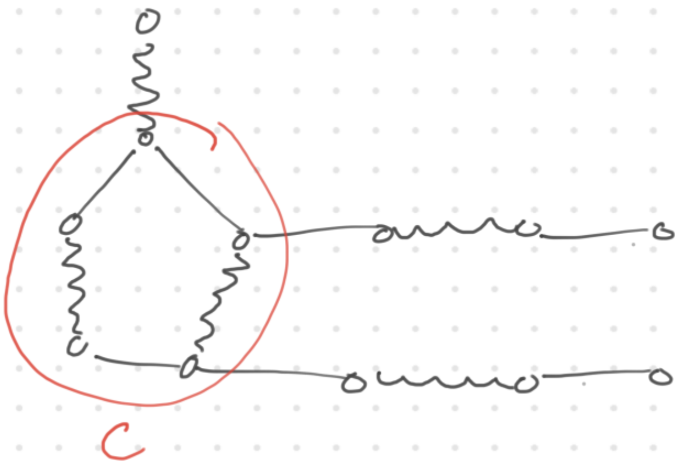
i - $(M - E(C))$-augmenting path in G/C

ii - we determine that $M - E(C)$ is maximum in G/C

Proposition shows that if we get outcome i, then we can find an M-augmenting path in G as well.

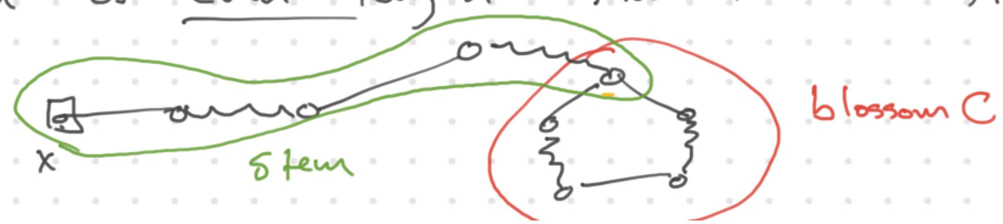
we'd like to say That if we get outcome ii, Then return "M is maximum in G"

Problem: ↑ is not true.



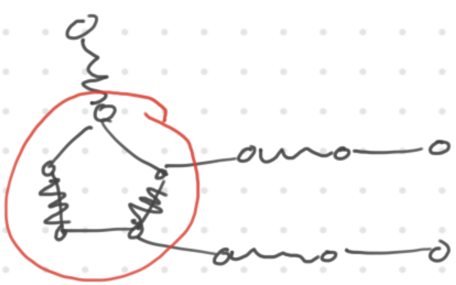
example of M in a graph G w/ blossom C ST $M - E(C)$ is a max matching in G/C

Def M a matching in a graph G + C a blossom + x an unmatched vertex, $x \notin V(C)$. A stem for blossom C with root x is an M-alternating (possibly 0) path of even length from $x \to V(C)$.



Prop Let C be an M-blossom w/ stem P rooted at x. Let $G' = G/C$. If $\exists$ an ~~an~~ M-augmenting path in G w/ x as an endpoint $\Rightarrow$ $\exists$ an $(M - E(C))$-augmenting path in G'.

This avoids the bad situation from before

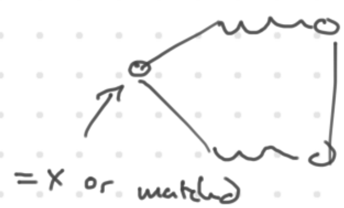


a blossom C st ~~G~~ ~~G~~ M is not maximum in G but $M - E(C)$ is maximum in G/C

Note: This blossom does not have a stem.

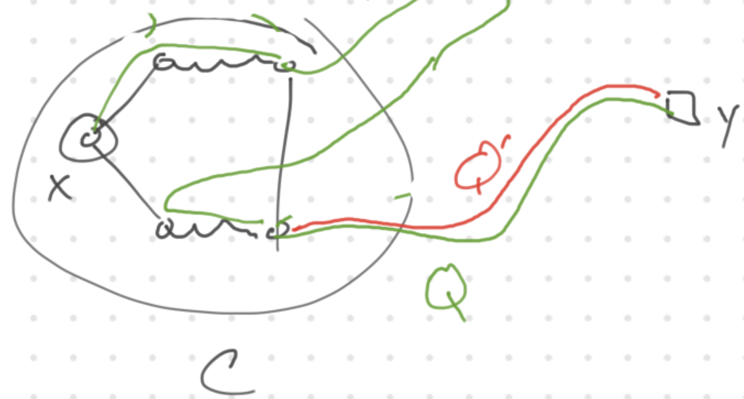
pf Let Q be an M-augmenting path in G w/ x as an endpoint, + let y be the other endpoint.

Cl $y \in V(C)$



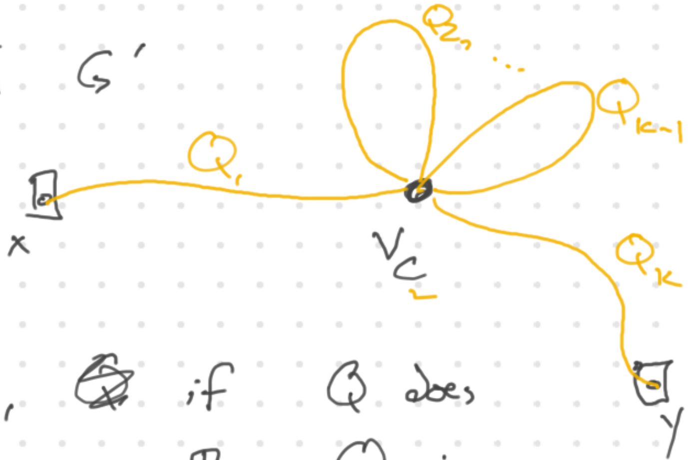
either $x \in V(C)$ or every vertex of C is matched + therefore there is no vertex of C ~~which~~ distinct from x which is unmatched $\Rightarrow$ $y \notin V(C)$. ✓

C1 wma $x \notin V(C)$



because x is not matched, in $G' = G/C$, every edge incident v_C is <u>not</u> in $E(M) - E(C)$ ie v_C is unmatched in G' $\Rightarrow$ subpath of Q from $y \to v_C$ (call it Q') is an $(M - E(C))$-augmenting path in G'

In G'



in G', if Q does not use v_C, then Q is an augmenting path from $x \to y$ + we're done so wma $Q \cap C \neq \emptyset$

Then Q in G' decomposes into $Q_1, Q_2, \ldots, Q_k$ where Q_1 is an $x \to v_C$ path, Q_k is a $v_C - y$ path + $Q_2, \ldots, Q_{k-1}$ are cycles intersecting <u>just</u> at v_C

Q_i's are pairwise edge disjoint + intersect exactly at the vertex v_c

The vertex v_c has exactly one matching edge incident to it. (it can't be 0 because of the matching edge in the stem)

if we were lucky, Q_1 would enter v_c on a matching edge + then $Q_1 \cup Q_k$

is an ~~M~~ $M-E(C))$-augmenting path in G' but that may not happen.

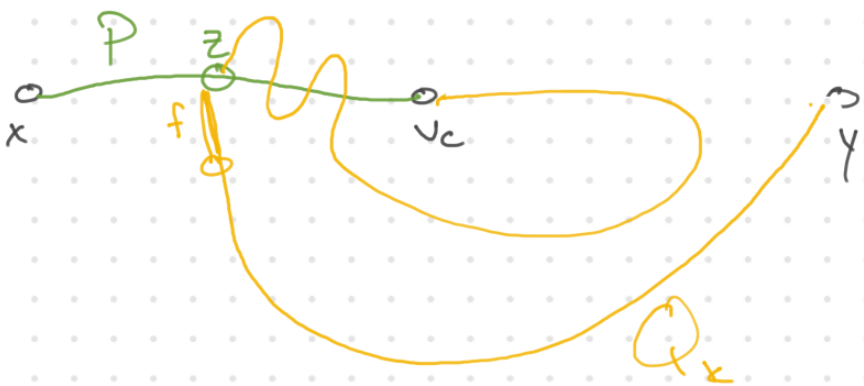
Assume that from all possible stems, we pick P to minimize $|E(P) \cup E(Q)|$

P is a path in G' from $x \to v_c$ entering on a matching edge

so if $Q_k \cap P = v_c$

$\Rightarrow$ $P \cup Q_k$ is an aug-path from $x \to y$,

$\Rightarrow$ wma Q_k intersects P at some internal vertex.



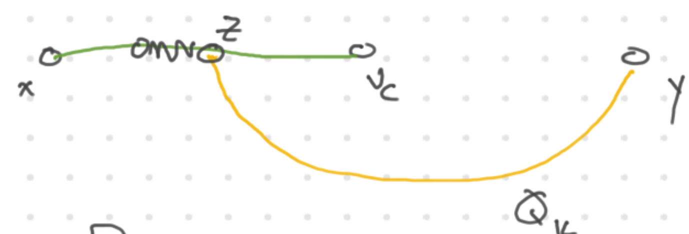
traversing Q_k from $y \to v_c$, let z be the first vertex of P we encounter

if f be the last edge of yQ_kz before hitting z, then $f \notin \mathcal{M}$ because

P is $\mathcal{M}$-alternating + so $\exists$ an $\mathcal{M}$ edge already in P which is incident to z.

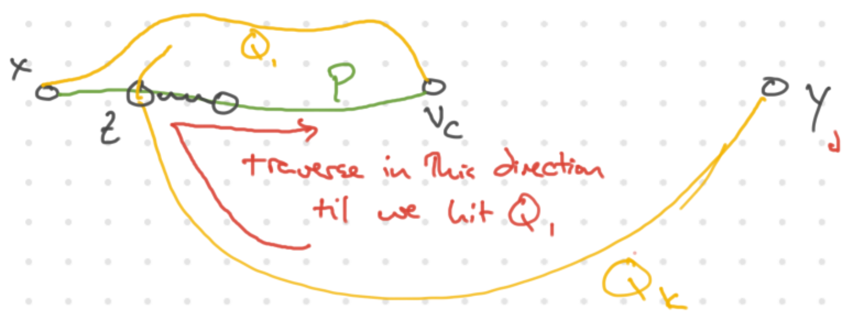
The $\mathcal{M}$-edge of P incident to z is either "left" (on xPz) or "right" (on zPv_c)

If $\exists$ an $\mathcal{M}$ edge on xPz incident to z (



$\Rightarrow$ $xPz \cup zQ_ky$ is $\mathcal{M}$-augmenting ✓

So wanna

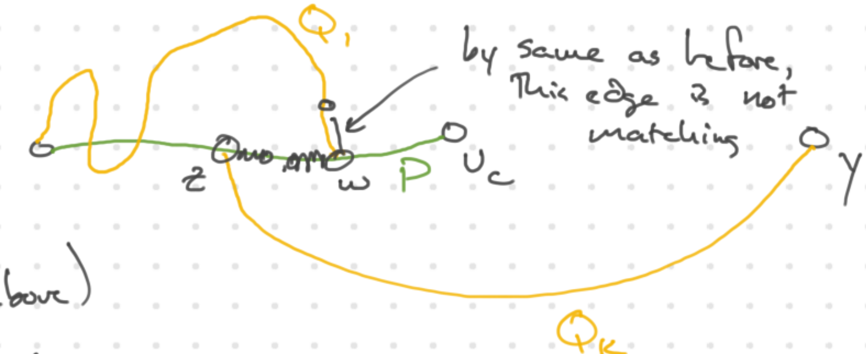


let w be the first vertex of Q_1 we encounter traversing zPv_c from $z \to v_c$

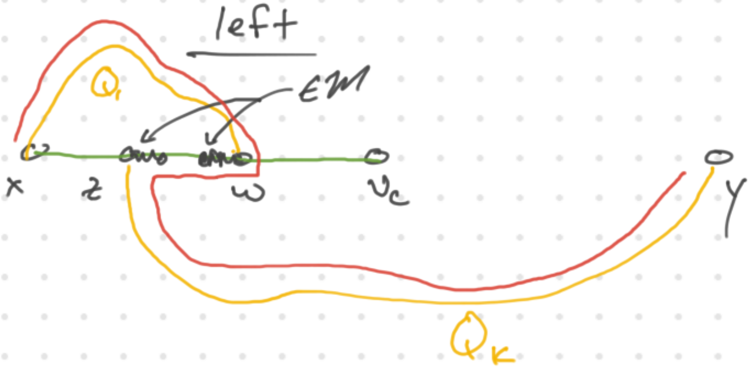
if $w = v_c$ (exactly the picture we drew above)
Then $Q_1 \cup zPv_c \cup zQ_k y$ is an augmenting path in G' (note P enters v_c on a matching edge by def of stem,

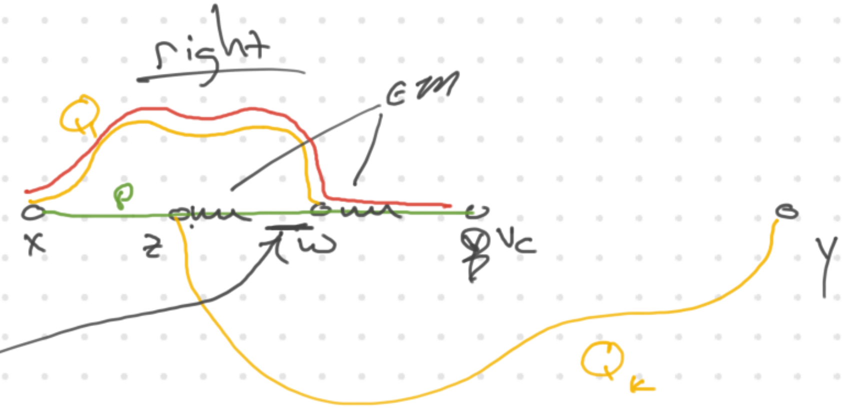
& therefore, Q_1 enters on a non-matching edge, so the path is still alternating at v_c)

Therefore, we hit Q_1 at a vertex w internal on zPv_c



Now matching edge incident to w is either left or right either the edge of $\mathcal{M} \cap P$ incident to w is "left" or "right"





we have an augmenting path from $x \to y$

$y Q_k z \cup z P w \cup w Q_1 x$

remember, we picked P to minimize $E(P) \cup E(Q)$. Replace P w/

$x Q_1 w \cup w P v_c$

This edge was in $P \cup Q$ & is no longer, we got rid of at least one edge of $P \cup Q$, $\rightarrow\leftarrow$ ∎